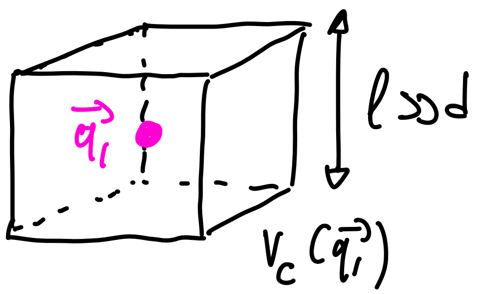
$$\partial_t f_1 + \{f_1, H_1\} = \int d\vec{q}_2\, d\vec{p}_2\, \frac{\partial V(\vec{q}_1 - \vec{q}_2)}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} f_2(\vec{q}_1, \vec{p}_1, \vec{q}_2, \vec{p}_2) \qquad (F_1)$$

Start from (F1) & build a coarse grained description over scales τ, ℓ such that

$$\tau_{MFP} \gg \tau \gg \tau_{col} \quad \& \quad \ell_{MFP} \gg \ell \gg d$$



$$\hat{f}_1(\vec{q}_1, \vec{p}_1, t) = \frac{1}{V_c} \int_{V_c(\vec{q}_1)} d^3\vec{q}\, f_1(\vec{q}, \vec{p}_1, t)$$

$$\partial_t f_1 + \{f_1, H_1\} \xrightarrow{\frac{1}{\tau V_c}\int_{V_c} d^3\vec{q} \int_t^{t+\tau} dt} \partial_t \hat{f}_1 + \{\hat{f}_1, H_1\}$$

Collisions $\quad d\Gamma = V_c(\vec{q}_1)\, d^3\vec{p}_1$

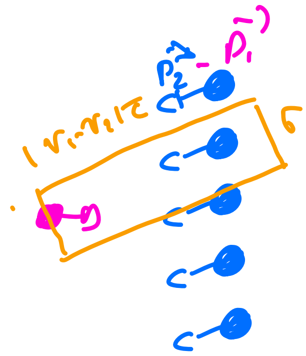
$$\underbrace{d\Gamma}_{\text{to go from density to numbers}} \int_t^{t+\tau} ds\, \partial_t \hat{f}_1\big|_{col} = N^+ - N^-$$

where N^+ = average number entering $d\Gamma$ due to a collision &

N^- = ——— leaving ———.

Loss term N^-: $\vec p_1, \vec p_2 \to \vec p_1{}', \vec p_2{}'$

$$N^- = \underbrace{\hat f_1(\vec q_1, \vec p_1, t)\, V_c\, d^3\vec p_1}_{\#\text{ of part with } \vec q_1, \vec p_1} \times \underbrace{\tau}_{\text{duration}} \times \underbrace{R^-(\vec p_1)}_{\text{rate of collision with other particles}}$$



\# of collision

$$\underbrace{\tau R^-(\vec p_1)}_{\#\text{ of collision}} = \int d^3\vec p_2 \; \underbrace{|\vec v_2 - \vec v_1|\, \tau\, \sigma(\vec p_2, \vec p_1)}_{\text{volume of particles that can collide with part \# 1}} \; \hat f_1(\vec q_1, \vec p_2, t)$$

\# of particles in the phase space volume

σ is the cross section

for hard spheres $\sigma = \pi d^2$.

attractive interactions, $\sigma > \pi d^2$.

$$\boxed{N^- = \hat f_1(\vec q_1, \vec p_1, t)\, V_c\, d^3\vec p_1\, \tau \int d^3\vec p_2\, |\vec v_1 - \vec v_2|\, \hat f_1(\vec q_1, \vec p_2, t)\, \sigma(\vec p_1, \vec p_2)}$$

Note that a collision between $\vec p_1$ & $\vec p_2$ may lead to a variety of $\vec p_1{}'$ & $\vec p_2{}'$.

Conservation of momentum & energy

$$\vec P_{cm}{}' = \frac{1}{2}(\vec p_1{}' + \vec p_2{}') = \vec P_{cm}$$

$$\vec p_d{}' = \frac{1}{2}(\vec p_1{}' - \vec p_2{}') = Rot(\theta, \varphi)\, \vec p_d$$

$$\sigma(\vec p_1, \vec p_2) = \int_{\theta, \varphi} d^2\sigma(\vec p_1, \vec p_2 \to \vec p_1{}', \vec p_2{}')$$

Gain term N^+:

We now need to consider collisions that lead to $\vec{p}_1, \vec{p}_2$, from $\vec{p}_1', \vec{p}_2'$.

We first note that

$$N^- = \int dN^- = \int_{\vec{p}_2, \theta, \varphi} dN(\vec{p}_2, \vec{p}_1 \to \vec{p}_2', \vec{p}_1')$$

Similarly we can define

$$dN^+ = dN(\vec{p}_2', \vec{p}_1' \to \vec{p}_2, \vec{p}_1)$$
$$= V_c \tau \, d^3\vec{p}_1' d^3\vec{p}_2' \, d^2\sigma(\vec{p}_1', \vec{p}_2' \to \vec{p}_1, \vec{p}_2) \, \hat{f}_1(\vec{p}_1', \vec{q}_1) \hat{f}_1(\vec{p}_2', \vec{q}_1) |\vec{v}_1' - \vec{v}_2'| \quad (1)$$

Can we put this in a form where it is more easily compared with dN^-?

* Time reversal symmetry

If $\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2'$ solves Hamilton's eq., so does $-\vec{p}_1', -\vec{p}_2' \to -\vec{p}_1, -\vec{p}_2$

* Parity symmetry (rotate by π in the $(\vec{p}_1, \vec{p}_2)$ plane)

If $\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2'$ solves Hamilton's eq., so does $-\vec{p}_1, -\vec{p}_2 \to -\vec{p}_1', -\vec{p}_2'$

$$\Rightarrow d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1', \vec{p}_2') \overset{TRS}{=} d^2\sigma(-\vec{p}_1', -\vec{p}_2' \to -\vec{p}_1, -\vec{p}_2) \overset{PS}{=} d^2\sigma(\vec{p}_1', \vec{p}_2' \to \vec{p}_1, \vec{p}_2) \quad (2)$$

* Collision = rotation of $\vec{p}_d = \frac{\vec{p}_1 - \vec{p}_2}{2}$

(i) $|\vec{v}_1' - \vec{v}_2'| = |\vec{v}_1 - \vec{v}_2|$ (3)

(ii) $\vec{p}_1 = \vec{p}_d + \vec{p}_{cm}$; $\vec{p}_2 = \vec{p}_{cm} - \vec{p}_d$; $K = \begin{vmatrix} \frac{d\vec{p}_1}{d\vec{p}_{cm}} & \frac{d\vec{p}_1}{d\vec{p}_d} \\ \frac{d\vec{p}_2}{d\vec{p}_{cm}} & \frac{d\vec{p}_2}{d\vec{p}_d} \end{vmatrix}$

$$d\vec{p}_1\, d\vec{p}_2 = k\, d\vec{p}_{CM}\, d\vec{p}_d = k\, d\vec{p}_{CM}{}'\, d\vec{p}_d{}' = d\vec{p}_1{}'\, d\vec{p}_2{}' \quad (4)$$

$\vec{p}_{CM} = \vec{p}_{CM}{}'$

$\vec{p}_d = Rot(\theta,\varphi).\vec{p}_d{}'$

Explicit computation:

$$\vec{p}_1{}' = \frac{1}{2}(\vec{p}_1+\vec{p}_2) + Rot.\frac{1}{2}(\vec{p}_1-\vec{p}_2) = \frac{Id+Rot}{2}\vec{p}_1 + \frac{Id-Rot}{2}\vec{p}_2$$

$$\vec{p}_2{}' = \frac{1}{2}(\vec{p}_1+\vec{p}_2) - Rot.\frac{1}{2}(\vec{p}_1-\vec{p}_2) = \frac{Id-Rot}{2}\vec{p}_1 + \frac{Id+Rot}{2}\vec{p}_2$$

$$\text{Jacobian: } 2^{-6}\begin{vmatrix} Id+Rot & Id-Rot \\ Id-Rot & Id+Rot \end{vmatrix}_{C_1\ \ C_2} = 2^{-6}\begin{vmatrix} Id+Rot & 2Id \\ Id-Rot & 2Id \end{vmatrix}_{C_1\ \ C_1+C_2} \begin{matrix} L_1 \\ L_2 \end{matrix}$$

$$= 2^{-6}\begin{vmatrix} 2Rot & 0 \\ Id-Rot & 2Id \end{vmatrix} \begin{matrix} L_1-L_2 \\ L_2 \end{matrix} = 2^{-6}.2^{6} = 1$$

Injecting (2), (3), (4) into (1) leads to

$$dN^{+} = V_c \tau\, d^3\vec{p}_1\, d^3\vec{p}_2\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, \hat{f}_1(\vec{p}_1{}',\vec{q}_1)\, \hat{f}_1(\vec{p}_2{}',\vec{q}_1)\, |\vec{v}_1-\vec{v}_2|$$

$$N^{+} = V_c \tau\, d^3\vec{p}_1 \underbrace{\int d^3\vec{p}_2\, d^2\sigma}_{}(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, \hat{f}_1(\vec{p}_1{}',\vec{q}_1)\, \hat{f}_1(\vec{p}_2{}',\vec{q}_1)\, |\vec{v}_1-\vec{v}_2|$$

Sum over all $\vec{p}_2, \vec{p}_1{}', \vec{p}_2{}'$ such that $\vec{p}_1,\vec{p}_2 \leftrightarrows \vec{p}_1{}',\vec{p}_2{}'$ can happen

Thus $\displaystyle\int_t^{t+\tau} ds \left.\frac{\partial \hat{f}_1}{\partial t}\right|_{col} = \frac{N^+ - N^-}{V_c\, d^3\vec{p}_1} = \tau \int d^3\vec{p}_2\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1 - \vec{v}_2| \Big[\hat{f}_1(\vec{p}_1{}',\vec{q}_1)\hat{f}_1(\vec{p}_1{}',\vec{q}_2) - \hat{f}_1(\vec{p}_1,\vec{q}_1)\hat{f}_1(\vec{p}_2,\vec{q}_2)\Big]$

All in all, this leads to the celebrated Boltzmann equation

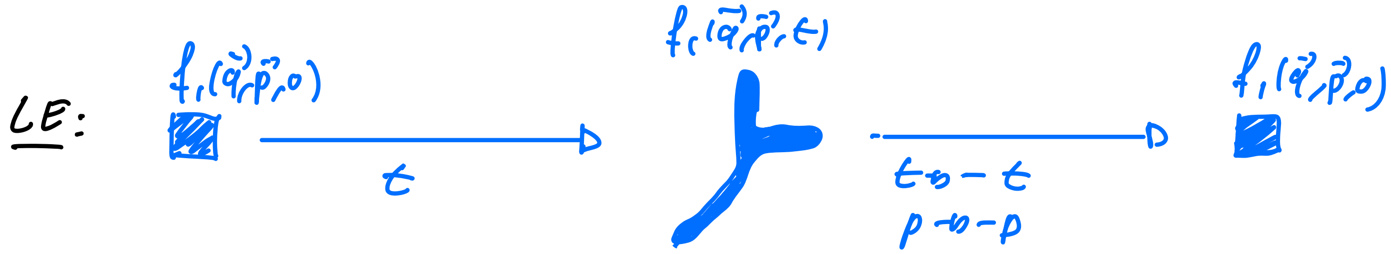
$$\frac{\partial \hat{f}_1(\vec{q}_1,\vec{p}_1,t)}{\partial t} + \{\hat{f}_1, H_1\} = \int d^3\vec{p}_2\, d^2\sigma\, |\vec{v}_1 - \vec{v}_2| \Big[\hat{f}_1(\vec{p}_1{}',\vec{q}_1)\hat{f}_1(\vec{p}_1{}',\vec{q}_2) - \hat{f}_1(\vec{q}_1,\vec{p}_1)\hat{f}_1(\vec{q}_1,\vec{p}_2)\Big] \qquad \text{(BE)}$$

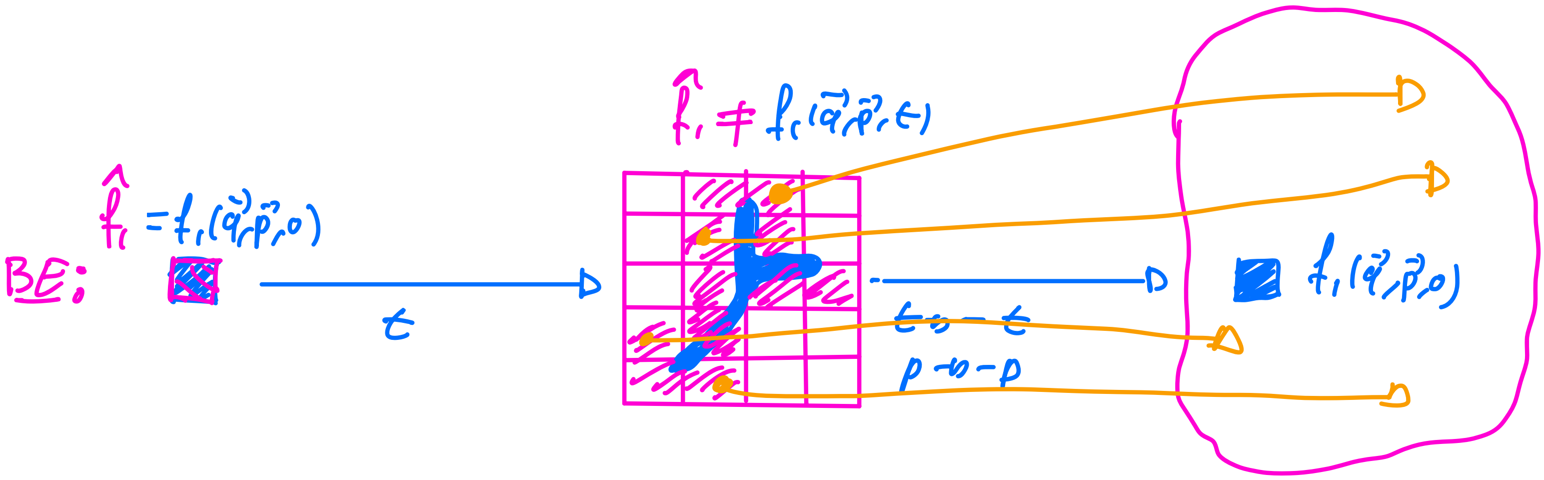
which is a closed equation for $\hat{f}_1$!

2.3) The H theorem

Let us show that if $\hat{f}_1$ solves (BE), then it relaxes irreversibly towards $\hat{f}_1^{eq} = \frac{N}{Z} e^{-\beta H_1(\vec{q}_1,\vec{p}_1)}$

Q: How is it possible since Liouville's equation is reversible?





Due to the coarse-graining, the fraction of initial conditions that are compatible with coming back is much less than 1.

How do we show this? Boltzmann H theorem.

Theorem: Let f be a solution of the Boltzmann equation, then $H(t) = \int d\vec{p}\, d\vec{q}\; f(\vec{q},\vec{p},t) \ln f(\vec{q},\vec{p},t)$ is a decreasing function of time.

Notation: For clarity, we drop the "hat": $\hat{f}_1 \to f_1$; the "1": $f_1 \to f$. We also write $f(\vec{p})$ as a proxy for $f(\vec{q},\vec{p})$.

Finaly, $f(\vec{p}_1)$ & $f(\vec{p}_2)$ are denoted f_1 & f_2, respectively,
and $f(\vec{p}_1{}')$ & $f(\vec{p}_2{}')$ ———— f_1' & f_2', respectively.

BE: $\partial_t f + \{f, H_1\} = \int d^3\vec{p}_2 \, d^2\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}') \, |\vec{v}_1 - \vec{v}_2| \, (f_1' f_2' - f_1 f_2)$

Proof: $\frac{d}{dt} H(t) = \int \underbrace{d\vec{p}\, d\vec{q}}_{d\Gamma} \, \partial_t [f \ln f] = \int d\vec{p}\, d\vec{q} \, (\ln f + 1) \, \partial_t f$

$$= \underbrace{\int d\Gamma \ln f \, \partial_t f}_{\equiv G} + \underbrace{\partial_t \Big(\underbrace{\int d\Gamma f}_{=N} \Big)}_{=0}$$

$G = \int d\Gamma \, \mathcal{O}(f) \, \partial_t f$ with $\mathcal{O}(f) = \ln f$

Using BE: (IBPs)

$$G = \underbrace{\int d\Gamma \ln f \left[\frac{\partial H}{\partial \vec{q}} \cdot \frac{\partial f}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{q}} \right]}_{①} + \underbrace{\int d\vec{p}_1 \, d\vec{p}_2 \, d\vec{q}_1 \, d^2\sigma \, |\vec{v}_1 - \vec{v}_2| \ln(f_1) (f_1' f_2' - f_1 f_2)}_{②}$$

① $= -\int d\Gamma \frac{1}{f} \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} f - \frac{1}{f} \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} f = -\int d\Gamma \frac{\partial}{\partial \vec{p}} \cdot \left[f \frac{\partial H}{\partial \vec{q}} \right] - \frac{\partial}{\partial \vec{q}} \cdot \left[f \frac{\partial H}{\partial \vec{p}} \right]$

$= 0$

In ②, $\vec{p}_1$ & $\vec{p}_2$ dummy variables so that

② $= \frac{1}{2}$ ② $+ \frac{1}{2}$ ② $(\vec{p}_1 \leftrightarrow \vec{p}_2)$

$$② = \frac{1}{2}\int d\vec{p}_1\, d\vec{p}_2\, d\vec{q}_1\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1-\vec{v}_2|\,(f_1' f_2' - f_1 f_2)\left[\ln f_1 + \ln f_2\right]$$

We also know that

$$d\vec{p}_1\, d\vec{p}_2 = d\vec{p}_1{}'\, d\vec{p}_2{}'$$

$$|\vec{v}_1 - \vec{v}_2| = |\vec{v}_1{}' - \vec{v}_2{}'|$$

$$d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}') = d^2\sigma(\vec{p}_1{}',\vec{p}_2{}' \to \vec{p}_1,\vec{p}_2)$$

so that

$$② = \frac{1}{2}\int d\vec{p}_1{}'\, d\vec{p}_2{}'\, d\vec{q}_1\, d^2\sigma(\vec{p}_1{}',\vec{p}_2{}' \to \vec{p}_1,\vec{p}_2)\, |\vec{v}_1{}'-\vec{v}_2{}'|\,(f_1' f_2' - f_1 f_2)\left[\ln f_1 + \ln f_2\right] \quad (5)$$

renaming $\vec{p}_1{}'$ & $\vec{p}_2{}'$ as $\tilde{\vec{p}}_1$ & $\tilde{\vec{p}}_2$, we see that $\vec{p}_1, \vec{p}_2 = \tilde{\vec{p}}_1{}', \tilde{\vec{p}}_2{}'$ since they are the image of $\tilde{\vec{p}}_1$ & $\tilde{\vec{p}}_2$ after a collision. Since $\tilde{\vec{p}}_1$ & $\tilde{\vec{p}}_2$ are dummy variables, we can drop the $\sim$ to get

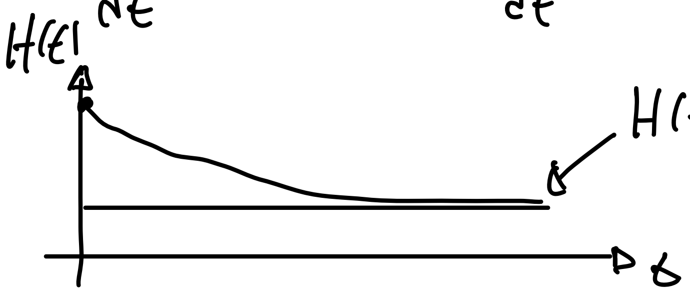
$$② = \frac{1}{2}\int d\vec{p}_1\, d\vec{p}_2\, d\vec{q}_1\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1-\vec{v}_2|\,(f_1 f_2 - f_1' f_2')\left[\ln f_1' + \ln f_2'\right] \quad (6)$$

$$② = \frac{1}{2}\left[(5)+(6)\right]$$

$$= \frac{1}{4}\int d\vec{p}_1\, d\vec{p}_2\, d\vec{q}_1\, d^2\sigma(\vec{p}_1,\vec{p}_2 \to \vec{p}_1{}',\vec{p}_2{}')\, |\vec{v}_1-\vec{v}_2|\,\underbrace{(f_1' f_2' - f_1 f_2)}_{\alpha}\underbrace{\left[\ln f_1 f_2 - \ln f_1' f_2'\right]}_{\beta} \leq 0$$

α and β have opposite signs

$$\Rightarrow \frac{d}{dt}H(t) \leq 0 \quad \& \quad \frac{d}{dt}H(t) = 0 \ \text{if} \ f_1' f_2' = f_1 f_2$$



H(t) is monotonously decreasing.

Q: What is the steady state?